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'Doubled' generalized Landau–Lifshitz hierarchies and special quasigraded Lie algebras

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Abstract

Using special quasigraded Lie algebras we obtain new hierarchies of integrable nonlinear vector equations admitting zero-curvature representations. Among them the most interesting is an extension of the generalized Landau–Lifshitz hierarchy called the 'doubled' generalized Landau–Lifshitz hierarchy. This hierarchy can also be interpreted as an anisotropic vector generalization of 'modified' sine–Gordon hierarchy or as a very special vector generalization of $so(3)$ anisotropic chiral field hierarchy.

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1. Introduction

Integrability of equations of $(1+1)$-field theory and condensed matter physics is based on the possibility to represent them in the form of the so-called zero-curvature equations [3, 1]:

$$\frac{\partial U(x,t,\lambda)}{\partial t} - \frac{\partial V(x,t,\lambda)}{\partial x} + [U(x,t,\lambda), V(x,t,\lambda)] = 0. \qquad (1)$$

The most productive interpretation of zero-curvature equations is achieved (see [2, 7]) if one treats them as a consistency condition for a set of commuting Hamiltonian flows on a dual space to some infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ of matrix-valued function of λ written in the Euler–Arnold (generalized Lax) form:

$$\frac{\partial L(\lambda)}{\partial t_l} = \mathrm{ad}^*_{\nabla I_l(L(\lambda))} L(\lambda), \qquad \frac{\partial L(\lambda)}{\partial t_k} = \mathrm{ad}^*_{\nabla I_k(L(\lambda))} L(\lambda), \qquad (2)$$

where $L(\lambda) \in \widetilde{\mathfrak{g}}^*$ is the generic element of the dual space, $\nabla I_k(L(\lambda)) \in \widetilde{\mathfrak{g}}$ is the algebra-valued gradient of $I_k(L(\lambda))$ and the 'Hamiltonians' $I_k(L(\lambda)), I_l(L(\lambda))$ belong to the set mutually commuting with respect to the natural Lie–Poisson bracket functions on $\widetilde{\mathfrak{g}}^*$. The consistency condition of two commuting flows given by equations (2) yields equation (1) with

$U \equiv \nabla I_k, V \equiv \nabla I_l, t_k \equiv x, t_l \equiv t$. In such a way, we obtain a lot of equations in partial derivatives that are indexed by two commuting Hamiltonians I_k and I_l. The set of equations (1) with fixed index k and all indices l constitute the so-called 'integrable hierarchy'. Hence, in order to construct new integrable hierarchies in the framework of the described approach it is necessary to have some infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ possessing an infinite set of mutually commuting Hamiltonians $\{I_k\}$ on its dual space. The main method that provides such a set is the famous Kostant–Adler scheme and its extensions [6, 2]. The main ingredient of this scheme is the existence of the decomposition of the algebra $\widetilde{\mathfrak{g}}$ into the sum of two subalgebras: $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-$.

Although the above described approach was originally based on employing the graded loop algebras $L(\mathfrak{g}) = \mathfrak{g} \otimes P(\lambda, \lambda^{-1})$ [2, 7] that possess decompositions into sums of two subalgebras, in the papers [8, 9] it was shown that a special Lie algebra $\mathfrak{g}_{\mathcal{E}}$ living on the elliptic curve $\mathcal{E}$ also possesses the decomposition $\mathfrak{g}_{\mathcal{E}} = \mathfrak{g}^+_{\mathcal{E}} + \mathfrak{g}^-_{\mathcal{E}}$ and can be used in order to produce integrable systems. In our papers [13–15], we have generalized this construction onto the case of special quasigraded Lie algebras $\mathfrak{g}_{\mathcal{H}}$ living on the algebraic curve $\mathcal{H}$. With their help we have obtained new integrable Hamiltonian systems (both finite and infinite dimensional) [14, 15]. In papers [16–18], we gave a Lie algebraic explanation of our previous semi-geometric construction of the Lie algebras $\mathfrak{g}_{\mathcal{H}}$. More explicitly, we have constructed a family of quasigraded Lie algebras $\mathfrak{g}_A$ parametrized by some numerical matrices A, such that loop algebras $L(\mathfrak{g})$ correspond to the case $A \equiv 0$ and quasigraded Lie algebras $\mathfrak{g}_{\mathcal{H}}$ to the case $A \in \mathrm{Diag}(n)$.

In the present paper, we generalize construction of [16–18] introducing even larger family of quasigraded Lie algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$ numbered by two numerical matrices A_1 and A_2 to which Kostant–Adler scheme may be applied. A family of Lie algebras $\widetilde{\mathfrak{g}}_A$ (see [16–18]) is embedded into the family of Lie algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$ as the algebras $\widetilde{\mathfrak{g}}_{1,A}$. We show that three types of integrable hierarchies are associated with the Lie algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$: two small hierarchies are associated with algebras $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$ and a large hierarchy is associated with the Lie algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$. We show that in the case when both matrices A_i are degenerate, the algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$ and $\widetilde{\mathfrak{g}}_A$ are not isomorphic as quasigraded Lie algebras. This means that integrable hierarchies associated with $\widetilde{\mathfrak{g}}_{A_1,A_2}$ such that $\det A_i = 0$ are not equivalent to the integrable hierarchies associated with $\widetilde{\mathfrak{g}}_A$ (see [15, 17, 18]). Moreover, we show that when the matrices A_i have the same matrix rank the subalgebras $\widetilde{\mathfrak{g}}^+_{A_1,A_2}$ and $\widetilde{\mathfrak{g}}^-_{A_1,A_2}$ are isomorphic, and the corresponding integrable hierarchies are also equivalent. That is why a 'large' integrable hierarchy associated with the whole Lie algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$ could be viewed as the 'double' of integrable hierarchy associated with $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$. The 'doubling' consists in adding 'negative' flows and new dynamical variables to the integrable hierarchy associated with $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$.

We consider these hierarchies in the case $\mathfrak{g} = so(n)$ and rank $A_i = n - 1$ in detail. We show that the integrable hierarchy associated with $\widetilde{so(n)}^{\pm}_{A_1,A_2}$ coincides with the $(n-1)$-component vector generalization of the ordinary three-component Landau–Lifshitz hierarchy. For $n > 4$, this hierarchy was first obtained in [12] using the technique of 'dressing' and Lie algebra of formal power series. The simplest equation of this hierarchy has the form

$$\frac{\partial \vec{s}}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \vec{s}}{\partial x^2} + \frac{3}{2}\left(\frac{\partial \vec{s}}{\partial x}, \frac{\partial \vec{s}}{\partial x}\right)\vec{s}\right) + \frac{3}{2}(\vec{s}, J\vec{s})\frac{\partial \vec{s}}{\partial x} \tag{3}$$

where $\vec{s}$ is the $(n-1)$-component vector and the tensor of anisotropy J is expressed via A_1 and A_2.

The 'double' of the generalized Landau–Lifshitz hierarchy is the 'large' integrable hierarchy associated with $\widetilde{so(n)}_{A_1,A_2}$. It is the $2(n-1)$-component hierarchy of vector equations satisfying two additional scalar constraints. The simplest equation of this hierarchy coincides with the two $(n-1)$-component vector differential equations of the first order. We show that for these equations two scalar constraints are easily solved and as a result we obtain two nonlinear $(n-2)$-component vector equations of the following form:

$$\partial_{x_+} \vec{s}_- = (c_- - (\vec{s}_-, \vec{s}_-))^{1/2} \hat{J}^{1/2} \vec{s}_+ \tag{4}$$

$$\partial_{x_-} \vec{s}_+ = (c_+ - (\vec{s}_+, \vec{s}_+))^{1/2} \hat{J}^{-1/2} \vec{s}_- \tag{5}$$

where $\vec{s}_\pm$ are the $(n-2)$-component vectors, $c_\pm$ are the arbitrary constants and anisotropy matrix J is connected with matrices A_i in a simple way (see section 3.4).

Equations (4) and (5) are, in a sense, the 'first negative equation' or a 'first negative flow' of the generalized L–L hierarchy, $\vec{s}_+$ are $(n-2)$-independent components of the $(n-1)$-component vector of the dynamical variables: $\vec{s} = (s_1, \vec{s}_+)$, $x_+ \equiv x$ is the space coordinate and x_- is the first 'negative' time.

It is necessary to note that in the $n = 3$ case equations (4) and (5) are equivalent to the 'modified sine–Gordon' equation [19, 20] and in the $n = 4$ case to the $so(3)$ anisotropic chiral field equations [21].

The structure of this paper is as follows: in section 2 we introduce algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$ and describe their properties. In section 3 we obtain integrable hierarchies associated with the Lie algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$ and their subalgebras $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$. In subsection 3.4 we consider the examples of this construction: the generalized Landau–Lifshitz hierarchy and its 'double'.

2. K–A admissible quasigraded Lie algebras

2.1. Lie algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$

Let $\mathfrak{g}$ be a classical matrix Lie algebra of the type $gl(n)$, $so(n)$ and $sp(n)$ over the field of the complex or real numbers. We will realize the algebra $so(n)$ as the algebra of skew-symmetric matrices: $so(n) = \{X \in gl(n) | X = -X^\top\}$ and the algebra $sp(n)$ as the following matrix algebra: $sp(n) = \{X \in gl(n) | X = sX^\top s\}$, where n is an even number, $s \in so(n)$ and $s^2 = -1$.

Let us introduce the new Lie bracket into the loop space $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$:

$$[X(\lambda), Y(\lambda)] = [X(\lambda), Y(\lambda)]_{A_1} - \lambda [X(\lambda), Y(\lambda)]_{A_2} \tag{6}$$

where $X(\lambda), Y(\lambda) \in \mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$, $\mathrm{Pol}(\lambda, \lambda^{-1})$ is the associative algebra of polynomial functions in λ and λ^{-1}, A_i are the numerical $n \times n$ matrices and $[X, Y]_{A_i} = XA_iY - YA_iX$.

Brackets $[X, Y]_{A_i} = XA_iY - YA_iX$ have arisen in the theory of consistent Poisson brackets on the finite-dimensional Lie algebras $\mathfrak{g}$ [10, 11]. In this paper, we use them in order to construct a new Lie bracket on the infinite-dimensional space $\mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$ (see also [16]).

The following proposition holds.

Proposition 2.1. *Let the numerical $n \times n$ matrices A_i, $i = 1, 2$ have the following form:*

(1) A_i is arbitrary for $\mathfrak{g} = gl(n)$,
(2) $A_i = A_i^\top$ for $\mathfrak{g} = so(n)$,
(3) $A_i = -sA_i^\top s$ for $\mathfrak{g} = sp(n)$.

Then bracket (6) is a correctly defined Lie bracket on $\mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$.

Remark 1. Matrices A_1 and A_2 are subjected only to the conditions of proposition 2.1 and are arbitrary otherwise. In particular, they may not commute with each other.

Definition. *We will denote infinite-dimensional space* $\mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$ *with the Lie bracket given by (6) by* $\widetilde{\mathfrak{g}}_{A_1,A_2}$ *and finite-dimensional Lie algebra* $\mathfrak{g}$ *with the bracket* $[\,,\,]_{A_i}$ *by* $\mathfrak{g}_{A_i}$

Remark 2. Algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$ could also be realized in the space of special matrix-valued functions of λ with an ordinary Lie bracket $[\,,\,]$. Nevertheless, we consider realization in the space $\mathfrak{g} \otimes \mathrm{Pol}(\lambda, \lambda^{-1})$ with the bracket (6) to be the most convenient.

Now we can introduce the convenient bases in the algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$. Because we are dealing with matrix Lie algebras $\mathfrak{g}$, we will denote their basic elements as X_{ij}. Let $X_{ij}^m \equiv X_{ij} \otimes \lambda^m$ be the natural basis in $\widetilde{\mathfrak{g}}_{A_1,A_2}$. Commutation relations (6) in this basis have the following form:

$$\left[X_{ij}^r, X_{kl}^m\right] = \sum_{p,q} C_{ij,kl}^{pq}(A_1) X_{pq}^{r+m} - \sum_{p,q} C_{ij,kl}^{pq}(A_2) X_{pq}^{r+m+1}, \tag{7}$$

where $C_{ij,kl}^{pq}(A_i)$ are the structure constants of the Lie algebras $\mathfrak{g}_{A_i}$.

Remark 3. Note that contrary to the case of loop algebras our algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$ admit only one type of decomposition $\widetilde{\mathfrak{g}}_{A_1,A_2} = \widetilde{\mathfrak{g}}_{A_1,A_2}^{+} + \widetilde{\mathfrak{g}}_{A_1,A_2}^{-}$ compatible with quasigrading, where the subalgebras $\widetilde{\mathfrak{g}}_{A_1,A_2}^{\pm}$ are defined in the natural way:

$$\widetilde{\mathfrak{g}}_{A_1,A_2}^{+} = \mathrm{Span}_{\mathbb{K}}\left\{X_{ij}^m \middle| m \geqslant 0\right\}, \qquad \widetilde{\mathfrak{g}}_{A_1,A_2}^{-} = \mathrm{Span}_{\mathbb{K}}\left\{X_{ij}^m \middle| m < 0\right\}. \tag{8}$$

Let us now find equivalences among the constructed Lie algebras. In particular, let us find conditions when $\widetilde{\mathfrak{g}}_{A_1,A_2}$ is equivalent to the algebra $\widetilde{\mathfrak{g}}_A \equiv \widetilde{\mathfrak{g}}_{1,A}$ introduced in our previous papers [16–18]. All equivalences are understood in the sense of the isomorphisms of quasigraded Lie algebras. The following proposition is true.

Proposition 2.2.

(i) The following isomorphisms hold: $\widetilde{\mathfrak{g}}_{A_1,A_2}^{\pm} \simeq \widetilde{\mathfrak{g}}_{A_2,A_1}^{\mp}$, $\widetilde{\mathfrak{g}}_{A_1,A_2} \simeq \widetilde{\mathfrak{g}}_{A_2,A_1}$.

(ii) If there exists matrix C such that $CA_1C = A_2$ *and* $C^2 = 1$ *then* $\widetilde{\mathfrak{g}}_{A_1,A_2}^{\pm} \simeq \widetilde{\mathfrak{g}}_{A_1,A_2}^{\mp}$.

(iii) If $\det A_1 \neq 0$ *or* $\det A_2 \neq 0$ *then* $\widetilde{\mathfrak{g}}_{A_1,A_2} \simeq \widetilde{\mathfrak{g}}_A$.

Remark 4. Item (iii) of the proposition means that in order for the algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$ not to be equivalent to the algebra $\widetilde{\mathfrak{g}}_A$ of [16, 18] matrices A_1 and A_2 should be degenerate. That is why we will consider the case $\det A_i = 0$ as the main case in this paper.

2.2. Coadjoint representation and its invariants

In this subsection, we define dual spaces, coadjoint representations and their invariants for the Lie algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$. At first, we explicitly describe the dual space $\widetilde{\mathfrak{g}}_{A_1,A_2}^{*}$ of $\widetilde{\mathfrak{g}}_{A_1,A_2}$. For this purpose, we define the pairing between $\widetilde{\mathfrak{g}}_{A_1,A_2}$ and $\widetilde{\mathfrak{g}}_{A_1,A_2}^{*}$ in the following way:

$$\langle X, L\rangle = \mathrm{res}_{\lambda=0} \mathrm{Tr}(X(\lambda)L(\lambda)). \tag{9}$$

The generic element of the dual space $\widetilde{\mathfrak{g}}_{A_1,A_2}^{*}$ with respect to this pairing is written as follows:

$$L(\lambda) = \sum_{k \in Z} \sum_{i,j=1,n} l_{ij}^{(k)} \lambda^{-(k+1)} X_{ij}^{*}, \tag{10}$$

where $l_{ij}^{(m)}$ are coordinate functions on $\widetilde{\mathfrak{g}}_{A_1,A_2}^{*}$. From the explicit form of the adjoint representation (6) and the pairing (9) it is easy to show that the coadjoint action of $\widetilde{\mathfrak{g}}_{A_1,A_2}$ on $\widetilde{\mathfrak{g}}_{A_1,A_2}^{*}$ has the form

$$\mathrm{ad}_{X(\lambda)}^{*} \circ L(\lambda) = \mathcal{A}(\lambda)X(\lambda)L(\lambda) - L(\lambda)X(\lambda)\mathcal{A}(\lambda), \tag{11}$$

where $X(\lambda), Y(\lambda) \in \widetilde{\mathfrak{g}}_{A_1,A_2}$, $L(\lambda) \in \widetilde{\mathfrak{g}}_{A_1,A_2}^{*}$, $\mathcal{A}(\lambda) = A_1 - \lambda A_2$.

Having the explicit form of the coadjoint action it is easy to deduce the next proposition:

Proposition 2.3. *Let* $L(\lambda)$ *be the generic element of* $\widetilde{\mathfrak{g}}^*_{A_1,A_2}$. *Then functions*

$$I_k^m(L(\lambda)) = (1/m)\,\mathrm{res}_{\lambda=0}\,\lambda^{-(k+1)}\,\mathrm{Tr}(L(\lambda)\mathcal{A}(\lambda)^{-1})^m \tag{12}$$

are invariants of the coadjoint representation of the algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$.

Remark 5. From the definition of the invariant functions it follows that in order to make the algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$ possess infinitely many algebraically independent invariants of coadjoint representation matrix $\mathcal{A}(\lambda)$ should be nondegenerate. This condition imposes additional requirements on the matrices A_i.

Remark 6. In the case of nondegenerate matrices A_i expression $\mathcal{A}(\lambda)^{-1}$ in formula (12) can be understood as the formal power series in the neighbourhood of zero or infinity:

$$\mathcal{A}(\lambda)^{-1} = \left(1 + A_1^{-1}A_2\lambda + \cdots\right)A_1^{-1} \qquad \text{or} \qquad \mathcal{A}(\lambda)^{-1} = -\lambda^{-1}A_2^{-1}\left(1 + A_1A_2^{-1}\lambda^{-1} + \cdots\right). \tag{13}$$

In the case of degenerate matrices A_i (but nondegenerate $\mathcal{A}(\lambda)$) one may still use formulae (12) and (13) considering matrices A_i as the limiting cases of some nondegenerate matrices $\mathcal{A}_i$ and then taking the corresponding limit in the suitably regularized expression for I_k^m. Another approach is to consider instead of integrals I_k^m an equivalent family of integrals given by the formula

$$I_k^{m'}(L(\lambda)) = (1/m)\,\mathrm{res}_{\lambda=0}\,\lambda^{-(k+1)}\,\mathrm{det}^m\,\mathcal{A}(\lambda)(\mathrm{Tr}(L(\lambda)\mathcal{A}(\lambda)^{-1})^m \tag{14}$$

which have no singularities for the degenerate A_i and yield the same algebra of invariants. Subsequently we will use the first approach as more convenient.

2.3. Lie–Poisson structure

Let us introduce Poisson structure in the space $\widetilde{\mathfrak{g}}^*_{A_1,A_2}$ using the above defined pairing $\langle\, , \rangle$. It defines a Lie–Poisson (Kirillov–Kostant) bracket on $P\left(\widetilde{\mathfrak{g}}^*_{A_1,A_2}\right)$ in the following standard way:

$$\{f_1(L(\lambda)), f_2(L(\lambda))\} = \langle L(\lambda), [\nabla f_1(L(\lambda)), \nabla f_2(L(\lambda))]_{\mathcal{A}(\lambda)}\rangle, \tag{15}$$

where

$$\nabla f_s(L(\lambda)) = \sum_{k\in Z}\sum_{i,j=1}^{n}\frac{\partial f_s}{\partial l_{ij}^{(k)}}X_{ij}^k, \; [\nabla f_1, \nabla f_2]_{\mathcal{A}(\lambda)} \equiv [\nabla f_1, \nabla f_2]_{A_1} - \lambda[\nabla f_1, \nabla f_2]_{A_2}.$$

From proposition 12 and standard considerations the next statement follows.

Proposition 2.4. *Functions* $I_k^m(L(\lambda))$ *are central for the Lie–Poisson bracket (15).*

Let us explicitly calculate the Poisson bracket (15). It is easy to show that for the coordinate functions $l_{ij}^{(m)}$ these brackets will have the following form:

$$\left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} = \sum_{p,q}C_{ij,kl}^{pq}(A_1)l_{pq}^{(n+m)} - \sum_{p,q}C_{ij,kl}^{pq}(A_2)l_{pq}^{(n+m+1)}. \tag{16}$$

Lie bracket (16) determines the structure of the Lie algebra isomorphic to $\widetilde{\mathfrak{g}}_{A_1,A_2}$ in the space of linear functions $\left\{l_{ij}^n\right\}$. That is why the corresponding Poisson algebra possesses a decomposition into the direct sum of two Poisson subalgebras, or in other words subspaces $\left(\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}\right)^*$ are Poisson.

3. Integrable hierarchies associated with algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$

In this section, we construct two infinite sets of mutually Poisson-commuting functions on the Lie algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$ and Lax-type representation for the corresponding Hamiltonian equations. We also derive zero-curvature equations as a compatibility condition of the above commuting Hamiltonian flows and consider examples of the equations in partial derivatives from the corresponding integrable hierarchies.

3.1. Infinite-component Hamiltonian systems on $\widetilde{\mathfrak{g}}^*_{A_1,A_2}$

In this subsection, we construct Hamiltonian systems on the infinite-dimensional space $\widetilde{\mathfrak{g}}^*_{A_1,A_2}$ possessing infinite number of independent, mutually commuting integrals of motion.

Let $L^{\pm}(\lambda)$ be the generic element of the space $\widetilde{\mathfrak{g}}^{\mp*}_{A_1,A_2}$:

$$L^{\pm}(\lambda) \equiv \sum_{i,j=1,n} L^{\pm}_{ij}(\lambda) X_{ji} = \sum_{k\in\mathbb{Z}_{\mp}} \sum_{i,j=1,n} l^{(k)}_{ij} \lambda^{-(k+1)} X_{ji}.$$

Let us consider the restriction of the invariant functions $\left\{I^m_k(L(\lambda))\right\}$ onto these subspaces. Note that although Poisson subspaces $\widetilde{\mathfrak{g}}^{\mp*}_{A_1,A_2}$ are infinite dimensional, functions $\left\{I^m_k(L^{\pm}(\lambda))\right\}$ are polynomials, i.e., after the restriction onto $\widetilde{\mathfrak{g}}^{\mp*}_{A_1,A_2}$ no infinite sums appear in their explicit expressions. Let us now consider functions $I^m_k(L^{\pm}(\lambda))$ as functions on the whole space $\widetilde{\mathfrak{g}}^*_{A_1,A_2}$. We have two sets of Hamiltonians $\left\{I^{m+}_k(L(\lambda))\right\}$ and $\left\{I^{m-}_k(L(\lambda))\right\}$ on $\widetilde{\mathfrak{g}}^*_{A_1,A_2}$ defined as follows:

$$I^{m\pm}_k(L(\lambda)) \equiv I^m_k(L^{\pm}(\lambda)).$$

Hamiltonian flows corresponding to Hamiltonians $I^{m\pm}_k(L(\lambda))$ are written in a standard way:

$$\frac{\partial L_{ij}(\lambda)}{\partial t^{m\pm}_k} = \left\{L_{ij}(\lambda), I^{m\pm}_k(L(\lambda))\right\}. \tag{17}$$

The following theorem is true.

Theorem 3.1.

(i) Hamiltonian equations (17) are written in the generalized Lax form:

$$\frac{\partial L(\lambda)}{\partial t^{m\pm}_k} = \mathrm{ad}^*_{V^{m\pm}_k(\lambda)} L(\lambda) = \mathcal{A}(\lambda) V^{m\pm}_k(\lambda) L(\lambda) - L(\lambda) V^{m\pm}_k(\lambda) \mathcal{A}(\lambda), \tag{18}$$

where $V^{m\pm}_k(\lambda) = \nabla I^{m\pm}_k(L(\lambda)) \equiv \sum_{s\in\mathbb{Z}_{\pm}} \sum_{i,j=1}^{n} \frac{\partial I^{m\pm}_k}{\partial l^{(s)}_{ij}} X^s_{ij}$.

(ii) The functions $\left\{I^{m\pm}_k(L(\lambda))\right\}$ *form the commutative subalgebra in the algebra of polynomial functions on* $\widetilde{\mathfrak{g}}^*_{A_1,A_2}$*:* $\left\{I^{m\pm}_k(L(\lambda)), I^{n\pm}_l(L(\lambda))\right\} = \left\{I^{m\mp}_k(L(\lambda)), I^{n\pm}_l(L(\lambda))\right\} = 0$, *i.e., time flows defined by equations (17) (or (18)) mutually commute.*

(iii) The functions $I^{n\pm}_l(L(\lambda))$ *are constant along all time flows:* $\frac{\partial I^{n\pm}_l}{\partial t^{m\pm}_k} = \frac{\partial I^{n\pm}_l}{\partial t^{m\mp}_k} = 0$.

The proof of this theorem repeats the proof of the analogous theorem for the case of ordinary loop algebras (see [2] and references therein).

Remark 7. Because the subspaces $\left(\widetilde{\mathfrak{g}}^{\mp}_{A_1,A_2}\right)^*$ are Poisson equations (17) generated by the Hamiltonians $I^{m\pm}_k(L(\lambda))$, could be restricted onto them, i.e., it is correct to consider the following Hamiltonian equations:

$$\frac{\partial L^+_{ij}(\lambda)}{\partial t^{m+}_k} = \left\{L^+_{ij}(\lambda), I^m_k(L^+(\lambda))\right\}, \qquad \frac{\partial L^-_{ij}(\lambda)}{\partial t^{m-}_k} = \left\{L^-_{ij}(\lambda), I^m_k(L^-(\lambda))\right\}. \tag{19}$$

In particular, the following corollary of theorem 3.1 holds.

Corollary 3.1.

(i) Hamiltonian equations (19) are written in the generalized Lax form:

$$\frac{\partial L^{\pm}(\lambda)}{\partial t_k^{m\pm}} = \mathcal{A}(\lambda)V_k^{m\pm}(\lambda)L^{\pm}(\lambda) - L^{\pm}(\lambda)V_k^{m\pm}(\lambda)\mathcal{A}(\lambda), \tag{20}$$

where $V_k^{m\pm}(\lambda)$ is defined as in theorem 3.1.

(ii) The functions $\left\{I_k^m(L^{\pm}(\lambda))\right\}$ form a commutative subalgebra in the algebra of polynomial functions on $\left(\widetilde{\mathfrak{g}}^{\mp}_{A_1,A_2}\right)^*$ and corresponding time flows mutually commute.

(iii) The functions $\left\{I_k^m(L^{\pm}(\lambda))\right\}$ are constant along time flows (19).

In other words, theorem 3.1 provides us with three types of infinite-component Hamiltonian systems possessing infinite sets of commuting integrals of motion: two 'small' Hamiltonian subsystems on the subspaces $\left(\widetilde{\mathfrak{g}}^{\mp}_{A_1,A_2}\right)^*$ with the sets of integrals $\left\{I_l^{n\pm}\right\}$ and the 'large' Hamiltonian system on $\widetilde{\mathfrak{g}}^{*}_{A_1,A_2}$ with both sets of integrals $\left\{I_l^{n-}\right\}$ and $\left\{I_k^{m+}\right\}$.

From theorem 3.1 it also follows that due to the commutativity of time flows it makes sense to consider $L(\lambda)$ as functions of independent time variables t_k^{m+} and t_l^{n-} and consider simultaneously all equations (18) as a system of differential identities of the first order on all the coordinate functions $l_{ij}^{(s)}\left(t_k^{m+}, t_l^{n-}\right)$ that are true on the 'Liouville torus' which is the level set of the integrals of motion: $\left\{I_l^n(L^+)\right\} = c_l^{n+}, I_k^m(L^-) = c_k^{m-}$. From this system of differential identities of the first order one can extract some finite subsystems of differential identities on some finite subsets of the coordinate functions $l_{ij}^{(s)}$. These identities are the required *integrable equations in partial derivatives*. In order to have a systematic procedure for obtaining such equations from equations (18), it is better to consider equivalent system of equations instead of (18). These will be zero-curvature equations with values in $\widetilde{\mathfrak{g}}_{A_1,A_2}$.

3.2. Zero-curvature conditions associated with algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$

Considering the consistency conditions of the commuting Lax-type equations (18) it is possible to prove the following theorem.

Theorem 3.2. *Let infinite-dimensional Lie algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$, $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$, their dual spaces and polynomial Hamiltonians $I_k^m(L^{\pm}(\lambda))$, $I_l^n(L^{\pm}(\lambda))$ on them be defined as in previous sections. Then the system of consistent generalized Lax equations (18) is equivalent to the system of the 'deformed' zero-curvature equations:*

$$\frac{\partial \nabla I_k^m(L^{\pm}(\lambda))}{\partial t_l^{n\pm}} - \frac{\partial \nabla I_l^n(L^{\pm}(\lambda))}{\partial t_k^{m\pm}} + \left[\nabla I_k^m(L^{\pm}(\lambda)), \nabla I_l^n(L^{\pm}(\lambda))\right]_{\mathcal{A}(\lambda)} = 0, \tag{21}$$

$$\frac{\partial \nabla I_k^m(L^{\pm}(\lambda))}{\partial t_l^{n\mp}} - \frac{\partial \nabla I_l^n(L^{\mp}(\lambda))}{\partial t_k^{m\pm}} + \left[\nabla I_k^m(L^{\pm}(\lambda)), \nabla I_l^n(L^{\mp}(\lambda))\right]_{\mathcal{A}(\lambda)} = 0. \tag{22}$$

(Proof of this theorem repeats the proof of the analogous theorem for algebras $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ (see [15]).)

Remark 8. Using the above-mentioned realizations of $\widetilde{\mathfrak{g}}_{A_1,A_2}$ 'deformed' zero-curvature equations can be rewritten in the form of the standard zero-curvature equations, but in this case corresponding U–V pairs will be more complicated and we will work with zero-curvature equations in the 'deformed' form (21) and (22).

The above theorem gives us possibility to distinguish three types of hierarchies, connected with the algebras $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$ and $\widetilde{\mathfrak{g}}_{A_1,A_2}$. Integrable hierarchies associated with the subalgebras $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$ are described only by equations (21), while integrable hierarchies associated with $\widetilde{\mathfrak{g}}_{A_1,A_2}$ are described by both equations (21) and (22), reflecting the fact that we have in this case both positive and negative flows. In other words, integrable hierarchies associated with the algebras $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$ could be viewed as subhierarchies of the integrable hierarchy associated with algebra $\widetilde{\mathfrak{g}}_{A_1,A_2}$. Nevertheless they are completely self-contained and could be considered separately. In particular, they do not depend on a 'large' algebra in which we embed corresponding subalgebra $\widetilde{\mathfrak{g}}^{+}_{A_1,A_2}$ or $\widetilde{\mathfrak{g}}^{-}_{A_1,A_2}$.

For the case of integrable hierarchies, associated with algebras $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$, a choice of the one of the matrix gradients ∇I^m_k to be the U operator yields fixation of dynamical variables that coincide with its matrix elements. For the case of integrable systems, connected with the algebras $\widetilde{\mathfrak{g}}_{A_1,A_2}$ there are two types of Hamiltonians and two types of flows. That is why in this case the number of independent dynamical variables may be doubled: their role is played by the matrix elements of two U operators: $U_+ = \nabla I^m_k(L^+(\lambda))$ and $U_- = \nabla I^n_l(L^-(\lambda))$, where Hamiltonians $I^m_k(L^+(\lambda))$ and $I^n_l(L^-(\lambda))$ generate evolution with respect to 'times' x_+ and x_-—'space' flows of the hierarchies associated with subalgebras $\widetilde{\mathfrak{g}}^{\pm}_{A_1,A_2}$.

The number of the dynamical variables for the chosen integrable hierarchy coincide with the number of independent matrix elements of the $U_\pm$ operators, where $U_+ = \nabla I^m_k(L^+(\lambda))$ and $U_- = \nabla I^n_l(L^-(\lambda))$. In our case, when $I^m_k(L^\pm(\lambda))$ depends on the additional parameters (matrix elements of the matrices A_i) we may decrease the number of dynamical variables manipulated by these parameters (in particular tending some of them to zero). Hence, this provides us with a simple procedure for the reduction of the number of functional degrees of freedom. We will illustrate this in the next subsection on the $\mathfrak{g} = so(n)$ example.

3.3. *Integrable subhierarchy associated with subalgebra* $\widetilde{so(n)}^{\pm}_{A_1,A_2}$

The aim of this subsection is a derivation of the equations of integrable hierarchy connected with the algebra $\widetilde{\mathfrak{g}}^{+}_{A_1,A_2}$, where $\mathfrak{g} = so(n)$, matrices A_i are degenerate: $\det A_i = 0$ and rank $A_i = n-1$. We will start our consideration with the nondegenerate case: rank $A_i = n$ and obtain the case rank $A_i = n-1$ as its limit.

Let us now illustrate the procedure of obtaining integrable equations in the partial derivatives starting from the Lie algebras $\widetilde{\mathfrak{g}}^{+}_{A_1,A_2}$, where $\mathfrak{g}$ and A_i are as described above. For this purpose, we have to describe the set of commuting integrals on $\big(\widetilde{so(n)}^{+}_{A_1,A_2}\big)^*$. Let us first note that the generic element of the dual space $\big(\widetilde{so(n)}^{+}_{A_1,A_2}\big)^*$ has the following form:

$$L^-(\lambda) = \lambda^{-1}L^{(0)} + \lambda^{-2}L^{(1)} + \lambda^{-3}L^{(2)} + \lambda^{-4}L^{(3)} + \cdots, \tag{23}$$

where $L^{(k)} \equiv \sum_{i<j=1,n} l^{(k)}_{ij} X_{ji}$. We will be interested in second-order integrals (Hamiltonians). By very definition they are written as follows:

$$I^{2-}_k(L(\lambda)) = \tfrac{1}{2}\operatorname{res}_{\lambda=0} \lambda^{-(k+1)} \operatorname{Tr}(L^-(\lambda)\mathcal{A}(\lambda)^{-1})^2. \tag{24}$$

In order for Hamiltonians I^{2-}_k to be polynomials we use the decomposition of the matrix $\mathcal{A}(\lambda)^{-1}$ in the formal power series in a neighbourhood of infinity:

$$I^{2-}(L(\lambda)) = \operatorname{Tr}\big((\lambda^{-1}L^{(0)} + \lambda^{-2}L^{(1)} + \cdots)\big(1 + A_1A_2^{-1}\lambda^{-1} + \cdots\big)A_2^{-1}\lambda^{-1}\big)^2. \tag{25}$$

The commuting integrals of the series $I^2(L^-(\lambda))$ contain the expression A_2^{-1} and in the limit $\det A_2 = 0$ should be regularized in the appropriate way. We will calculate these Hamiltonians for the case $\det A_2 \neq 0$ and then consider the limit $\det A_2 \to 0$.

The simplest Hamiltonians of the set (25) are the functions $I^{2-}_{-4}(L(\lambda))$ and $I^{2-}_{-5}(L(\lambda))$:[1]

$$\begin{aligned} I^{2-}_{-4}(L(\lambda)) &= \tfrac{1}{2}\,\mathrm{Tr}\left(A_2^{-1}L^{(0)}\right)^2, \\ I^{2-}_{-5}(L(\lambda)) &= \mathrm{Tr}\left(A_2^{-1}L^{(0)}A_1A_2^{-2}L^{(0)}\right) + \left(A_2^{-1}L^{(1)}A_2^{-1}L^{(0)}\right). \end{aligned} \tag{26}$$

We will hereafter consider the case $A_1 = \mathrm{diag}\big(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}\big)$, $A_2 = \mathrm{diag}\big(a_1^{(2)}, a_2^{(2)}, \ldots, a_{n-1}^{(2)}, a_n^{(2)}\big)$ and take in the previous formulae limit $a_n^{(2)} \to 0$. Because Hamiltonians $I^{2-}_{-4}(L(\lambda))$ and $I^{2-}_{-5}(L(\lambda))$ are singular in this limit we have to rescale them, considering the limit $a_n^{(2)} \to 0$ of commuting integrals

$$I^{2-'}_{-4}(L(\lambda)) \equiv a_n^{(2)} I^{2-}_{-4}(L(\lambda)), \qquad I^{2-'}_{-5}(L(\lambda)) \equiv \left(\left(a_n^{(2)}/a_n^{(1)}\right)I^{2}_{-5}(L^-(\lambda)) - I^{2}_{-4}(L^-(\lambda))\right).$$

Taking this limit we obtain

$$\begin{aligned} I^{2-'}_{-4}(L(\lambda)) &= \sum_{i<n} \frac{\left(l_{in}^{(0)}\right)^2}{a_i^{(2)}}, \\ I^{2-'}_{-5}(L(\lambda)) &= \frac{1}{a_n^{(1)}} \sum_{i<n} \left(2\frac{l_{in}^{(1)} l_{in}^{(0)}}{a_i^{(2)}} + \frac{\left(l_{in}^{(0)}\right)^2 a_i^{(1)}}{\left(a_i^{(2)}\right)^2}\right) - \sum_{0<i<j<n} \frac{\left(l_{ij}^{(0)}\right)^2}{a_i^{(2)} a_j^{(2)}}. \end{aligned} \tag{27}$$

The corresponding matrix gradients are written as follows:

$$\begin{aligned} \frac{1}{2}\nabla I^{2-'}_{-4} &= \sum_{i<n} \frac{l_{in}^{(0)}}{a_i^{(2)}} X_{in}, \\ \frac{1}{2}\nabla I^{2-'}_{-5} &= \frac{1}{a_n^{(1)}} \sum_{i<n} \left(\lambda \frac{l_{in}^{(0)}}{a_i^{(2)}} X_{in} + \left(\frac{l_{in}^{(1)}}{a_i^{(2)}} + \frac{a_i^{(1)} l_{in}^{(0)}}{\left(a_i^{(2)}\right)^2}\right) X_{in}\right) - \sum_{i<j<n} \frac{l_{ij}^{(0)}}{a_i^{(2)} a_j^{(2)}} X_{ij}. \end{aligned} \tag{28}$$

Let us take for the Hamiltonian that generates a space flow the function $I^{2-'}_{-4}$. This fixes our integrable hierarchy with $U \equiv \nabla I^{2-'}_{-4}$. Taking into account the explicit form of $\nabla I^{2-'}_{-4}$ we obtain that the dynamical variables in the corresponding hierarchy are the functions $l_{in}^{(0)}, i \in \overline{1, n-1}$. Hence, by taking the limit $a_n^{(2)} \to 0$, we have decreased the number of functional degrees of freedom from $n(n-1)/2$ (number of the independent components of ∇I^{2-}_{-4}) to $n-1$ (number of the independent components of $\nabla I^{2-'}_{-4}$).

In order to obtain all equations of this hierarchy it is necessary to obtain the regularized expression $\nabla I^{m-'}_{k}$ for the all other Hamiltonians ∇I^{m-}_{k} to express all coordinate functions $l_{ij}^{(k)} k \geqslant 0$ via $l_{in}^{(0)}$ and its derivatives with respect to the coordinate x and substitute them into expression for $\nabla I^{m-'}_{k}$ into the zero-curvature equation:

$$\frac{\partial \nabla I^{2-'}_{-4}}{\partial t_k^{m'}} = \frac{\partial \nabla I^{m-'}_{k}}{\partial x} - \left[\nabla I^{2-'}_{-4}, \nabla I^{m-'}_{k}\right]_{A(\lambda)}. \tag{29}$$

We will consider the simplest equation of the hierarchy (29) that corresponds to the time flow of the Hamiltonian $I^{2-'}_{-5}$, i.e., we will put $V \equiv \nabla I^{2'}_{-5}$. The coordinate functions $l_{ij}^{(0)}$ and $l_{in}^{(1)}$, where $i, j \in \overline{1, n-1}$ enter in the explicit expression of the V operator (28). They should be expressed via $l_{in}^{(0)}$ and their derivatives in order to obtain a required equation on the dynamical variables $l_{in}^{(0)}$. This can be achieved by decomposing both sides of equation (29) in the powers of spectral parameter λ. Rescaling time variables $x \to 2x, t \to 2t$ and introducing the

[1] In the case of the nondegenerate matrices A_i corresponding matrix gradients produce 'integrable anisotropic deformation' of the generalized Heisenberg magnet equations [15].

following notation: $m_i^{(1)} = l_{in}^{(1)} + \frac{a_i^{(1)} l_{in}^{(0)}}{a_i^{(2)}}$, we obtain that for the chosen U–V pair zero-curvature equation (29) is equivalent to the following system of differential equations:

$$\frac{\partial l_{in}^{(0)}}{\partial t} - \frac{\partial m_i^{(1)}}{\partial x} = \sum_{k=1}^{n-1} \frac{l_{ik}^{(0)} a_k^{(1)} l_{kn}^{(0)}}{\left(a_k^{(2)}\right)^2}, \tag{30}$$

$$\frac{\partial l_{in}^{(0)}}{\partial x} = \sum_{k=1}^{n-1} \frac{l_{ik}^{(0)} l_{kn}^{(0)}}{a_k^{(2)}}, \tag{31}$$

$$\frac{\partial l_{ij}^{(0)}}{\partial x} = a_n^{(1)} \left(m_i^{(1)} l_{jn}^{(0)} - m_j^{(1)} l_{in}^{(0)}\right). \tag{32}$$

We will use equations (31) and (32) in order to express $m_i^{(1)}$ and $l_{ij}^{(0)}$ via dynamical variables $l_{jn}^{(0)}$ and their x derivatives. From these equations, it is easy to deduce that the following equalities hold:

$$l_{ij}^{(0)} = \frac{\partial l_{in}^{(0)}}{\partial x} l_{jn}^{(0)} - \frac{\partial l_{jn}^{(0)}}{\partial x} l_{in}^{(0)}, \tag{33}$$

$$m_i^{(1)} = 1/a_n^{(1)} \frac{\partial^2 l_{in}^{(0)}}{\partial x^2} + c_2(L^{(0)}) l_{jn}^{(0)}, \tag{34}$$

where $c_2(L^{(0)})$ is some scalar function of the dynamical variables $l_{in}^{(0)}$. We determine the explicit form of the function $c_2(L^{(0)})$ using the fact that Hamiltonians $I_{-4}^{2'}$ and $I_{-5}^{2'}$ are constant along all flows and we may put $I_{-4}^{2'} = 1, I_{-5}^{2'} = 0$. Using this and introducing vector $(\overrightarrow{l})_i = l_{in}^{(0)}/a_i^{(2)}$ and matrices $A_i' = \mathrm{diag}\left(a_1^{(i)}, a_2^{(i)}, a_3^{(1)}, \ldots, a_{n-1}^{(i)}\right)$ we obtain

$$c_2(\overrightarrow{l}) = \frac{1}{2}(\overrightarrow{l}, A_1' \overrightarrow{l}) + 1/a_n^{(1)} \cdot \frac{3}{2} \left(\frac{\partial \overrightarrow{l}}{\partial x}, A_2' \frac{\partial \overrightarrow{l}}{\partial x} \right).$$

Using equality (33) we also deduce that

$$\sum_{k=1}^{n-1} \frac{l_{ik}^{(0)} a_k^{(1)} l_{kn}^{(0)}}{a_i^{(2)} \left(a_k^{(2)}\right)^2} = -\frac{1}{2} \frac{\partial (\overrightarrow{l}, A_1' \overrightarrow{l})}{\partial x} (\overrightarrow{l})_i + (\overrightarrow{l}, A_1' \overrightarrow{l}) \frac{\partial (\overrightarrow{l})_i}{\partial x}.$$

As a result we obtain the following differential equation in the partial derivatives:

$$\frac{\partial \overrightarrow{l}}{\partial t} = \frac{1}{a_n^{(1)}} \frac{\partial}{\partial x} \left(\frac{\partial^2 \overrightarrow{l}}{\partial x^2} + \frac{3}{2} \left(\frac{\partial \overrightarrow{l}}{\partial x}, A_2' \frac{\partial \overrightarrow{l}}{\partial x} \right) \overrightarrow{l} \right) + \frac{3}{2} (\overrightarrow{l}, A_1' \overrightarrow{l}) \frac{\partial \overrightarrow{l}}{\partial x}. \tag{35}$$

In order to transform this equation to a more standard form it is necessary to introduce new notation: $\overrightarrow{s} = (A_2')^{1/2} \overrightarrow{l}$, $J \equiv A_1' (A_2')^{-1}$. Under such a replacement of variables the constraint $(\overrightarrow{l}, A_2' \overrightarrow{l}) = 1$ passes to the standard constraint $(\overrightarrow{s}, \overrightarrow{s}) = 1$ and equation (35) to the higher Landau–Lifshitz equation:

$$\frac{\partial \overrightarrow{s}}{\partial t} = \frac{1}{a_n^{(1)}} \frac{\partial}{\partial x} \left(\frac{\partial^2 \overrightarrow{s}}{\partial x^2} + \frac{3}{2} \left(\frac{\partial \overrightarrow{s}}{\partial x}, \frac{\partial \overrightarrow{s}}{\partial x} \right) \overrightarrow{s} \right) + \frac{3}{2} (\overrightarrow{s}, J \overrightarrow{s}) \frac{\partial \overrightarrow{s}}{\partial x}. \tag{36}$$

Remark 9. In the case $n = 4$, this equation is the higher equation of the Landau–Lifshitz hierarchy. For $n > 4$, this equation was first obtained in [12] using the technique of 'dressing' and the embedding of the (specially realized) algebra $\widetilde{so(n)}_A^+$ into algebra $so(n)(\lambda)$ of formal power series. Equation (36) was also obtained in our previous paper [18] using the algebra $\widetilde{so(n)}_A^+$ naturally embedded into algebra $\widetilde{so(n)}_A$.

In the next subsection, we will obtain the simplest equation of 'doubled' Landau–Lifshitz hierarchy. In order to do so it is necessary to employ neither the algebra of formal power series nor the algebra $\widetilde{so(n)}_A$ but its generalization—Lie algebra $\widetilde{so(n)}_{A_1,A_2}$.

3.4. Integrable hierarchy associated with algebra $\widetilde{so(n)}_{A_1,A_2}$

In this subsection, we will consider integrable hierarchies, admitting zero-curvature-type representation with U–V pairs taking values in the algebra $\widetilde{so(n)}_{A_1,A_2}$. In order to obtain 'double' of Landau–Lifshitz hierarchies it is necessary to consider the case of the matrix $\mathcal{A}(\lambda)$ formed by the degenerate matrices A_i, such that rank $A_i = n-1$ but rank $\mathcal{A}(\lambda) = n$. As in the previous example of the hierarchies connected with $\widetilde{so(n)}^{\pm}_{A_1,A_2}$ we will first consider the case of the nondegenerate matrices A_i: rank $A_i = n$ and obtain the case rank $A_i = n-1$ as its limit.

Let us now illustrate the procedure of obtaining integrable equations in the partial derivatives associated with algebras $\widetilde{so(n)}_{A_1,A_2}$. For this purpose, we have to describe the set of commuting integrals on $\big(\widetilde{so(n)}_{A_1,A_2}\big)^*$. The generic elements of the dual spaces to subalgebras $\big(\widetilde{so(n)}^-_{A_1,A_2}\big)^*$ and $\big(\widetilde{so(n)}^+_{A_1,A_2}\big)^*$ have the following form:

$$L^+(\lambda) = L^{(-1)} + \lambda L^{(-2)} + \lambda^2 L^{(-3)} + \lambda^3 L^{(-4)} + \cdots, \tag{37}$$

$$L^-(\lambda) = \lambda^{-1} L^{(0)} + \lambda^{-2} L^{(1)} + \lambda^{-3} L^{(2)} + \lambda^{-4} L^{(3)} + \cdots, \tag{38}$$

where $L^{(\pm k)} \equiv \sum_{i<j=1,n} l_{ij}^{(\pm k)} X_{ji}$. The second-order integrals (Hamiltonians) by very definition are written as follows:

$$I_k^{2\pm}(L(\lambda)) = \tfrac{1}{2}\,\mathrm{res}_{\lambda=0}\, \lambda^{-(k+1)}\, \mathrm{Tr}\,(L^{\pm}(\lambda)\mathcal{A}(\lambda)^{-1})^2. \tag{39}$$

Let us at first consider Hamiltonians $I_k^{2\pm}$ in the case of the nondegenerate matrices A. In order for Hamiltonians $I_k^{2\pm}$ to be polynomials we will use two different decompositions of the matrix $\mathcal{A}(\lambda)^{-1}$ in formal power series—in the neighbourhood of zero and infinity. The corresponding Hamiltonians are calculated using their own decompositions:

$$I_k^{2+}(L(\lambda)) = \tfrac{1}{2}\,\mathrm{res}_{\lambda=0}\, \lambda^{-(k+1)}\, \mathrm{Tr}\,\big(A_1^{-1}\big(1 + A_1^{-1}A_2\lambda + \cdots\big)(L^{(-1)} + \lambda L^{(-2)} + \cdots)\big)^2, \tag{40}$$

$$I_k^{2-}(L(\lambda)) = \tfrac{1}{2}\,\mathrm{res}_{\lambda=0}\, \lambda^{-(k+1)}\, \mathrm{Tr}\,\big(\big(1 + A_1A_2^{-1}\lambda^{-1} + \cdots\big)A_2^{-1}\lambda^{-1}(\lambda^{-1}L^{(0)} + \lambda^{-2}L^{(1)} + \cdots)\big)^2. \tag{41}$$

The simplest Hamiltonians of these sets are functions $I_{-4}^{2-}(L(\lambda))$ and $I_0^{2+}(L(\lambda))$:[2]

$$I_{-4}^{2-}(L(\lambda)) = \tfrac{1}{2}\,\mathrm{Tr}\,\big(A_2^{-1}L^{(0)}\big)^2, \qquad I_0^{2+}(L(\lambda)) = \tfrac{1}{2}\,\mathrm{Tr}\,\big(A_1^{-1}L^{(-1)}\big)^2. \tag{42}$$

Without a loss of generality we will assume that matrices A_i are diagonal: $A_1 = \mathrm{diag}\big(a_1^{(1)}, \ldots, a_n^{(1)}\big)$, $A_2 = \mathrm{diag}\big(a_1^{(2)}, \ldots, a_n^{(2)}\big)$ and consider the limits $a_1^{(1)} \to 0, a_n^{(2)} \to 0$ that correspond to the simplest degeneration of the matrices A_i. Because Hamiltonians I_{-4}^{2-} and I_0^{2+} are singular in this limit we will rescale them and consider integrals $a_n^{(2)} I_{-4}^{2-}$ and

[2] In the case of the nondegenerate matrices A_i the corresponding matrix gradients produce anisotropic chiral field-type equations [17].

$a_1^{(1)} I_0^{2+}$ instead. As a result we obtain the following Hamiltonians:

$$I_{-4}^{2-'}(L(\lambda)) \equiv \lim_{a_n^{(2)},\to 0} a_n^{(2)} I_{-4}^{2-} = \frac{1}{2}\sum_{i<n} \frac{\left(l_{in}^{(0)}\right)^2}{a_i^{(2)}},$$
$$I_0^{2+'}(L(\lambda)) \equiv \lim_{a_1^{(1)}\to 0} a_1^{(1)} I_0^{2+} = \frac{1}{2}\sum_{i>1} \frac{\left(l_{1i}^{(-1)}\right)^2}{a_i^{(1)}}. \tag{43}$$

Their matrix gradients are written as follows:

$$\nabla I_{-4}^{2-'} = \sum_{i<n} \frac{l_{in}^{(0)}}{a_i^{(2)}} X_{in}, \qquad \nabla I_0^{2+'} = \lambda^{-1} \sum_{i>1} \frac{l_{1i}^{(-1)}}{a_i^{(1)}} X_{1i}. \tag{44}$$

These are exactly U operators of two independent generalized Landau–Lifshitz hierarchies. That is why we call this hierarchy to be '*doubled*' *generalized Landau–Lifshitz hierarchy*.

The corresponding zero-curvature condition

$$\frac{\partial \nabla I_{-4}^{2-'}}{\partial x_-} - \frac{\partial \nabla I_0^{2+'}}{\partial x_+} + \left[\nabla I_{-4}^{2-'}, \nabla I_0^{2+'}\right]_{\mathcal{A}(\lambda)} = 0, \tag{45}$$

yields the following equations:

$$\partial_{x_+} l_{1i}^{(-1)} = -\left(a_i^{(1)}/a_i^{(2)}\right) l_{1n}^{(-1)} l_{in}^{(0)}, \qquad \partial_{x_-} l_{in}^{(0)} = -a_i^{(2)}/a_i^{(1)} l_{1n}^{(0)} l_{1i}^{(-1)}, \tag{46}$$

$$\partial_{x_+} l_{1n}^{(-1)} = a_n^{(1)} \sum_{k=2}^{n-1} l_{1k}^{(-1)} l_{kn}^{(0)} / a_k^{(2)}, \qquad \partial_{x_-} l_{1n}^{(0)} = a_1^{(2)} \sum_{k=2}^{n-1} l_{1k}^{(-1)} l_{kn}^{(0)} / a_k^{(1)}. \tag{47}$$

Taking into account that functions

$$I_0^{2+'}(L^+(\lambda)) = \frac{1}{2}\sum_{i>1} \frac{\left(l_{1i}^{(-1)}\right)^2}{a_i^{(1)}} = c_-, \qquad I_{-4}^{2-'}(L(\lambda)) = \frac{1}{2}\sum_{i=1}^{n-1} \frac{\left(l_{in}^{(0)}\right)^2}{a_i^{(2)}} = c_+$$

are constant along all time flows, we obtain that $l_{1n}^{(-1)}, l_{1n}^{(0)}$ are expressed via $l_{1i}^{(-1)}$ and $l_{in}^{(0)}$:

$$l_{1n}^{(-1)} = \left(a_n^{(1)}\right)^{1/2} \left(c_- - \sum_{i=2}^{n-1} \frac{\left(l_{1i}^{(-1)}\right)^2}{a_i^{(1)}}\right)^{1/2}, \qquad l_{1n}^{(0)} = \left(a_1^{(2)}\right)^{1/2} \left(c_+ - \sum_{i=2}^{n-1} \frac{\left(l_{in}^{(0)}\right)^2}{a_i^{(2)}}\right)^{1/2} \tag{48}$$

and equations (47) follows from equations (46).

Introducing for convenience the following $(n-2)$-component vectors:

$$s_-^i = \frac{l_{in}^{(-1)}}{\left(a_i^{(1)}\right)^{1/2}}, \qquad s_+^i = \frac{l_{in}^{(0)}}{\left(a_i^{(2)}\right)^{1/2}}, \qquad i \in 2, n-1,$$

and rescaling variables $x_\pm$ we obtain that our equations acquire the following form:

$$\partial_{x_+} \overrightarrow{s}_- = (c_- - (\overrightarrow{s}_-, \overrightarrow{s}_-))^{1/2} \widehat{J}^{1/2} \overrightarrow{s}_+, \tag{49}$$

$$\partial_{x_-} \overrightarrow{s}_+ = (c_+ - (\overrightarrow{s}_+, \overrightarrow{s}_+))^{1/2} \widehat{J}^{-1/2} \overrightarrow{s}_-, \tag{50}$$

where the $(n-2) \times (n-2)$ matrix $\widehat{J}$ is defined as follows: $\widehat{J} = \mathrm{diag}\left(\left(a_2^{(2)}\right)^{-1} a_2^{(1)}, \ldots, \left(a_{n-1}^{(2)}\right)^{-1} a_{n-1}^{(1)}\right)$.

Remark 10. Note that the variables $\overrightarrow{s}_-$ could be expressed via $\overrightarrow{s}_+$ and its derivatives with respect to 'negative time' x_- using equation (50). As a result one obtains a system of nonlinear differential equations of the second order on the vector $\overrightarrow{s}_+$. Such a procedure breaks the simple form of the obtained equations and we prefer to leave them in the form of the systems (49) and (50).

Let us now consider the small n example of equations (49) and (50).

Example 1. Let $n = 3$. In this case, we obtain the following two equations:

$$\partial_{x_+} s_- = (c_- - s_-^2)^{1/2} j^{1/2} s_+, \qquad \partial_{x_-} s_+ = \left(c_+ - s_+^2\right)^{1/2} j^{-1/2} s_-.$$

Making substitution of variables: $s_\pm = c_\pm \sin\phi_\pm$, and rescaling variables $x_\pm$ we obtain

$$\partial_{x_+}\phi_- = \sin\phi_+, \qquad \partial_{x_-}\phi_+ = \sin\phi_-.$$

Expressing ϕ_- via ϕ_+ and putting it into the first equation we finally obtain

$$\partial^2_{x_+x_-}\phi_+ = (1 - (\partial_{x_-}\phi_+)^2)^{1/2} \sin\phi_+. \tag{51}$$

This is exactly the so-called 'modified sine–Gordon equation' discovered by Kruskal and re-discovered later by Chen [19] (see also [20] and references therein).

Example 2. Let $n = 4$. In this case, equations (49) and (50) define 'the first negative flow' to standard Landau–Lifshitz equations. They have the following form:

$$\partial_{x_+} s^1{}_- = (c_- - ((s_-^1)^2 + (s_-^2)^2))^{1/2} J_1^{1/2} s^1{}_+, \qquad \partial_{x_-} s^1{}_+ = \left(c_+ - \left(\left(s_+^1\right)^2 + \left(s_+^2\right)^2\right)\right)^{1/2} J_1^{-1/2} s^1{}_-,$$

$$\partial_{x_+} s^2{}_- = (c_- - ((s_-^1)^2 + (s_-^2)^2))^{1/2} J_2^{1/2} s^2{}_+, \qquad \partial_{x_-} s^2{}_+ = \left(c_+ - \left(\left(s_+^1\right)^2 + \left(s_+^2\right)^2\right)\right)^{1/2} J_2^{-1/2} s^2{}_-.$$

Adding the third component to the vectors $\overrightarrow{s}_\pm$: $s_\pm^3 = \left(c_\pm - \left(\left(s_\pm^1\right)^2 + \left(s_\pm^2\right)^2\right)\right)^{1/2}$, rescaling one of the 'times' $x'_- = (J_1 J_2)^{-1/2} x_-$ and making the following change of indices in vector $\overrightarrow{s}_-$: $s_-^2 \longleftrightarrow s_-^1$ we obtain that the above equations are written as follows:

$$\partial_{x_+} s^1{}_- = s^3{}_- J_2^{1/2} s^2{}_+, \qquad \partial_{x'_-} s^1{}_+ = s^3{}_+ J_2^{1/2} s^2{}_-, \tag{52}$$

$$\partial_{x_+} s^2{}_- = s^3{}_- J_1^{1/2} s^1{}_+, \qquad \partial_{x'_-} s^2{}_+ = s^3{}_+ J_1^{1/2} s^1{}_-, \tag{53}$$

$$\partial_{x_+} s^3{}_- = -\left(J_2^{1/2} s^1{}_- s^2{}_+ + J_1^{1/2} s^2{}_- s^1{}_+\right), \qquad \partial_{x'_-} s^3{}_+ = -\left(J_2^{1/2} s^1{}_+ s^2{}_- + J_1^{1/2} s^2{}_+ s^1{}_-\right). \tag{54}$$

This system of equations coincides with the anisotropic chiral field equations of Cherednik [21]:

$$\frac{\partial \overrightarrow{s_-}}{\partial x_+} = [\overrightarrow{s_-} \times \widetilde{J}(\overrightarrow{s_+})], \qquad \frac{\partial \overrightarrow{s_+}}{\partial x'_-} = [\overrightarrow{s_+} \times \widetilde{J}(\overrightarrow{s_-})],$$

where the diagonal matrix $\widetilde{J}$ is defined as follows: $\widetilde{J} = \mathrm{diag}\left(J_1^{1/2}, -J_2^{1/2}, 0\right)$.

4. Conclusion and discussion

In this present paper, for all classical matrix Lie algebras $\mathfrak{g}$ we have constructed a family of quasigraded Lie algebras $\mathfrak{g}_{A_1,A_2}$ numbered by two numerical matrices A_1 and A_2 to which the Kostant–Adler scheme may be applied. Employing them we have obtained new hierarchies of integrable nonlinear vector equations admitting zero-curvature representations. In particular, we have obtained a vector generalization of the Landau–Lifshitz hierarchy and its extension with negative flows, called the 'doubled' Landau–Lifshitz hierarchy.

Let us make several comments about a possible development and generalization of the results of this paper. One of the simplest possibilities is to consider the constructed Lie algebras with poles at the points $\lambda = \nu_+, \lambda = \nu_-$ instead of the poles at the points $\lambda = 0$ and $\lambda = \infty$. Unfortunately, contrary to the case of ordinary loop algebras, change of the pole location will not lead to the appearance of new integrable hierarchies. Another, more perspective development is a search for more complicated quasigraded Lie algebras that admit the Kostant–Adler scheme. Work in this direction is now in progress.

Also a very interesting open problem consists in obtaining 'non-stationary' Hamiltonians and 'non-stationary' (i.e., x dependent) Lie–Poisson structures that yield the constructed PDEs as Hamiltonian equations in the field-theoretical sense. For the case of general n this problem is complicated and the answer is still not known to the author.

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